Polyhedra and Lattices

Louis-Noël Pouchet

CS & ECE
Colorado State University

February 23, 2020

PPoPP'20 Tutorial

Math Corner

The Affine Qualifier

Definition (Affine function)

A function $f : \mathbb{K}^m \rightarrow \mathbb{K}^n$ is affine if there exists a vector $\vec{b} \in \mathbb{K}^n$ and a matrix $A \in \mathbb{K}^{m \times n}$ such that:

$$\forall \vec{x} \in \mathbb{K}^m, \; f(\vec{x}) = A\vec{x} + \vec{b}$$

Definition (Affine half-space)

An affine half-space of $\mathbb{K}^m$ (affine constraint) is defined as the set of points:

$$\{\vec{x} \in \mathbb{K}^m \mid \vec{a}.\vec{x} \leq \vec{b}\}$$

Polyhedron (Implicit Representation)

Definition (Polyhedron)

A set $\mathcal{S} \in \mathbb{K}^m$ is a polyhedron if there exists a system of a finite number of inequalities $A\vec{x} \leq \vec{b}$ such that:

$$\mathcal{P} = \{\vec{x} \in \mathbb{K}^m \mid A\vec{x} \leq \vec{b}\}$$

Equivalently, it is the intersection of finitely many half-spaces.

Definition (Polytope)

A polytope is a bounded polyhedron.

Integer Polyhedron

Definition ($\mathbb{Z}$-polyhedron)

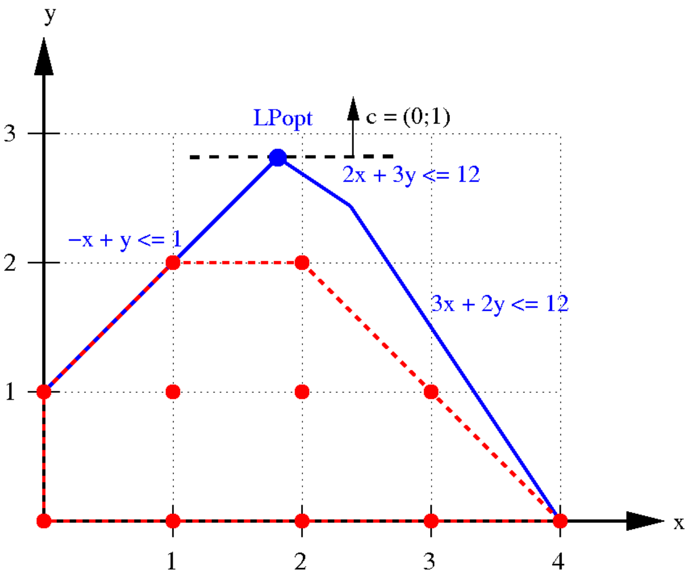
It is a polyhedron where all its extreme points are integer valued

Definition (Integer hull)

The integer hull of a rational polyhedron $\mathcal{P}$ is the largest set of integer points such that each of these points is in $\mathcal{P}$.

For the moment, we will "say" an integer polyhedron is a polyhedron of integer points (language abuse)

Rational and Integer Polytopes



Another View of Polyhedra

The dual representation models a polyhedron as a combination of lines L and rays R (forming the polyhedral cone) and vertices V (forming the polytope)

Definition (Dual representation)

$$\mathcal{P} : \{\vec{x} \in \mathbb{Q}^n \mid \vec{x} = L\vec{\lambda} + R\vec{\mu} + V\vec{\nu},\ \vec{\mu} \geq 0,\ \vec{\nu} \geq 0,\ \sum_i \nu_i = 1\}$$

Definition (Face)

A face $\mathcal{F}$ of $\mathcal{P}$ is the intersection of $\mathcal{P}$ with a supporting hyperplane of $\mathcal{P}$. We have:

$$dim(\mathcal{F}) \leq dim(\mathcal{P})$$

Definition (Facet)

A facet $\mathcal{F}$ of $\mathcal{P}$ is a face of $\mathcal{P}$ such that:

$$dim(\mathcal{F}) = dim(\mathcal{P}) - 1$$

Some Equivalence Properties

Theorem (Fundamental Theorem on Polyhedral Decomposition)

If $\mathcal{P}$ is a polyhedron, then it can be decomposed as a polytope $\mathcal{V}$ plus a polyhedral cone $\mathcal{L}$.

Theorem (Equivalence of Representations)

Every polyhedron has both an implicit and dual representation

- Chernikova's algorithm can compute the dual representation from the implicit one
- The Dual representation is heavily used in polyhedral compilation
- Some works operate on the constraint-based representation (Pluto)

Parametric Polyhedra

Definition (Paramteric polyhedron)

Given $\vec{n}$ the vector of symbolic parameters, $\mathcal{P}$ is a parametric polyhedron if it is defined by:

$$\mathcal{P} = \{\vec{x} \in \mathbb{K}^m \mid A\vec{x} \leq B\vec{n} + \vec{b}\}$$

- Requires to adapt theory and tools to parameters
- Can become nasty: case distinctions (QUAST)
- Reflects nicely the program **context**

Some Useful Algorithms

All extended to parametric polyhedra:

- Compute the facets of a polytope: **PolyLib** [Wilde et al]
- Compute the volume of a polytope (number of points): **Barvinok** [Claus/Verdoolaege]
- Scan a polytope (code generation): **CLooG** [Quillere/Bastoul]
- Find the lexicographic minimum: **PIP** [Feautrier]

Operations on Polyhedra

Definition (Intersection)

The intersection of two convex sets $\mathcal{P}_1$ and $\mathcal{P}_2$ is a convex set $\mathcal{P}$:

$$\mathcal{P} = \{\vec{x} \in \mathbb{K}^m \mid \vec{x} \in \mathcal{P}_1 \wedge \vec{x} \in \mathcal{P}_2\}$$

Definition (Union)

The union of two convex sets $\mathcal{P}_1$ and $\mathcal{P}_2$ is a set $\mathcal{P}$:

$$\mathcal{P} = \{\vec{x} \in \mathbb{K}^m \mid \vec{x} \in \mathcal{P}_1 \vee \vec{x} \in \mathcal{P}_2\}$$

The union of two convex sets may not be a convex set

Lattices

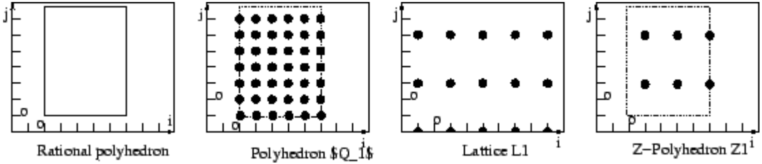
Definition (Lattice)

A subset L in $\mathbb{Q}^n$ is a lattice if is generated by integral combination of finitely many vectors: $a_1, a_2, \ldots, a_n$ ($a_i \in \mathbb{Q}^n$). If the a_i vectors have integral coordinates, L is an integer lattice.

Definition ($\mathcal{Z}$-polyhedron)

A $\mathcal{Z}$-polyhedron is the intersection of a polyhedron and an affine integral full dimensional lattice.

Pictured Example



Example of a $\mathcal{Z}$-polyhedron:

- $Q_1 = \{i, j \mid 0 \leq i \leq 5,\ 0 \leq 3j \leq 20\}$
- $L_1 = \{2i+1, 3j+5 \mid i, j \in \mathbb{Z}\}$
- $Z_1 = Q_1 \cap L_1$

Quick Facts on $\mathcal{Z}$-polyhedra

- **Iteration domains are in fact $\mathcal{Z}$-polyhedra with unit lattice** if loop has unit stride
- Intersection of $\mathcal{Z}$-polyhedra is not convex in general
- Union is complex to compute
- **Can count points, can optimize, can scan**

- **Implementation available for most operations in PolyLib**

Image and Preimage

Definition (Image)

The image of a polyhedron $\mathcal{P} \in \mathbb{Z}^n$ by an affine function $f : \mathbb{Z}^n \to \mathbb{Z}^m$ is a $\mathcal{Z}$-polyhedron $\mathcal{P}'$:

$$\mathcal{P}' = \{f(\vec{x}) \in \mathbb{Z}^m \mid \vec{x} \in \mathcal{P}\}$$

Definition (Preimage)

The preimage of a polyhedron $\mathcal{P} \in \mathbb{Z}^n$ by an affine function $f : \mathbb{Z}^n \to \mathbb{Z}^m$ is a $\mathcal{Z}$-polyhedron $\mathcal{P}'$:

$$\mathcal{P}' = \{\vec{x} \in \mathbb{Z}^n \mid f(\vec{x}) \in \mathcal{P}\}$$

We have $Image(f^{-1}, \mathcal{P}) = Preimage(f, \mathcal{P})$ if f is invertible.